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# On non-commutative $\ell^1$-algebra

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## 1 Introduction

The group algebra $L^1(G)$ for a locally compact abelian group $G$ has a long rich history in connection with Fourier analysis and theory of Banach algebras. On the other hand, in the theory of C*-algebras the concept of crossed products prevails for a long time as a basic tool for the analysis of the actions of groups to $C^\star$-algebras. In this context, we construct first a Banach *-algebra (viewed as a non-commutative $L^1$-algebra) and then obtain the $C^\star$-crossed product for a given group action as its $C^\star$-envelope. The simplest model for this context is the case $C(X)$ ,the algebra of all continuous functions on a compact Hausdorff space $X$, with a single automorphism $\alpha$ (the action of the integer group $Z$ on $C(X)$). We recall that this system comes from a dynamical system $\Sigma=\{X, \sigma\}$ for a compact space $X$ with a homeomorphism $\sigma$. In this case, such an algebra writing as $\ell^1(\Sigma)$ is defined as follows.

The algebra consists of $C(X)$- valued sequences, $a=(a(n))$ with its norm $\Sigma_Z\|a(n)\|$ and the product as convolutions twisted by the automorphism $\alpha$. That is, for $a=(a(n))$ and $b=(b(n))$ their product $c=ab=(c(n))$ is written as

$$c(n)=\Sigma_k a(k)\alpha^k(b(n-k)).$$

With the *-operation (involution) defined as $a^\star(n)=\alpha^n(\overline{a(-n)})$, the algebra $\ell^1(\Sigma)$ then becomes a Banach *-algebra with the isometric involution. We denote its $C^\star$-envelope,$C^\star$-crossed product, by $C^\star(\Sigma)$. Here when $X$ consists of just one point this algebra is nothing but the usual $\ell^1(Z)$, hence its $C^\star$-envelope is $C(T)$ on the torus $T$. Hence, as an algebra of continuous functions,we can see the simple structure of $C(T)$ fairly well. On the contrary, although we know the Gelfand representation of $l^1(Z)$ the structure of this algebra is much more complicated, notably shown by the existence of a non-selfadjoint closed ideal in $\ell^1(Z)$. This situation is similar to the impossibility of harmonic synthesis by Malliavin [4, Theorem 7.6.1]. Thus even in this trivial case there appears a serious difference between structures of $l^1(\Sigma)$ and that of $C^*(\Sigma)$. Unfortunately, almost no people have been paying attention to these non-commutative $L^1$ (or $\ell^1$) algebras, contrary to commutative $L^1$-algebras.

This lecture intends to initiate the study of these non-commutative $L^1$-algebras starting from $\ell^1(\Sigma)$.

# 2 Results

We fist mention the following

**Proposition 2.1** *The dual of $\ell^1(\Sigma)$ is linearly isometric to the Banach space $\ell^\infty(Z, C(X)^\star)$, that is, the space of all $C(X)^\star$- valued bounded sequences with the supreme norm.*

Here we note first that $C(X)$ ($C^\star$-algebra) is assumed to be a closed subalgebra of $\ell^1(\Sigma)$ by the embedding $f \in C(X) \to \hat{f} = (\hat{f}(n))$ where $\hat{f}(0) = f$ and $\hat{f}(n) = 0$ for all non-zero $n$.We identify this algebra with $C(X)$. The proof of this result then goes through along the similar line of the proof of the fact that the dual of $\ell^1(Z)$ is $\ell^\infty(Z)$, where we just replace complex numbers by the dual of $C(X)$.

Now this result suggests that though there are suficiently many states (positive functionals with norm one) on this Banach *-algebra leading its envelope $C^\star(\Sigma)$ in an injective way there may appear many functionals satisfying the conditions $\varphi(1) = 1 = \|\varphi\|$ but not states . In fact, for a functional $\varphi = \{\varphi_n\}$ this condition only concernes with the component $\varphi_0$ of the functional $\varphi$ and no restriction for other components $\varphi_n$ for $n \neq 0$ is imposed except of the relation $\|\varphi_n\| \leq 1$. Therefore, there are actually plenty of non-positive functionals on this $\ell^1$-algebra satisfying that required condition.

In connection with this we can show the next result. Recall the well known fact that for a unital $C^\star$-algebra a functional satisfying the above condition becomes necessarily a state, that is , becomes positive. On the other hand, the converse assertion has been remaining somewhat obscure.

**Proposition 2.2** *Let $A$ be a unital Banach *-algebra , then every functional $\varphi$ satisfying the above condition becomes a state if and only if $A$ becomes a $C^\star$-algebra with the same norm and the involution.*

Similar but a little different result is mentioned in [3, Theorem 11.2.5]. We however give here a direct proof together with related facts for readers convenience.

For a unital Banach algebra $B$ (without assuming an involution) the sets $S(B)$ and $Her(B)$ are defined as follows.

$$S(B) = \{\varphi \in B^\star | \quad \varphi(1) = 1 = ||\varphi||\},$$

$$Her(B) = \{a \in B \quad | \quad \varphi(a) \in R \quad \forall \varphi \in S(B)\}.$$

{Proof of the Proposiion}. It is enough to show for the converse that every selfadjoint element of $A$ belongs to $Her(A)$ because once we know this fact the rest is an immediate consequence of the following Vidaf-Palmer theorem [1, Theorem 38.14]. This theorem asserts that

" Suppose that $B$ is expressed as $B = Her(B) + iHer(B)$, then the map :$(h + ik)^\star = h - ik$ defines an involution of $B$, for which $B$ becomes a $C^\star$-algebra with respect to this involution".

Thus, we assert that if $a = h + ik$ for selfadjoint elements $h, k$ then $h$ and $k$ belong to $Her(A)$, so that the above theorem applies here. Now take a functional $\varphi$ in $S(B)$.By the assumption, we know that $S(A)$ is actually the set of states on $A$, but we may not assert apriori that the value $\varphi(h)$ as well as $\varphi(k)$ is real contrary to the case of $C^\star$-algebras. Now we may assume here that $||h|| < 1$. Then, we see by [6, Chap.1 Lemma 9.8] that there exists a selfadjoint element $b$ such that $1 - h = b^2$, that is, $1 - h$ is positive. It follows that $\varphi(1 - h) = 1 - \varphi(h) \geq 0$. Hence $\varphi(h)$ is real and $h$ belongs to $Her(A)$, so does $k$.

This proposition together with the first result shows in a sense how this non-commutative $\ell^1$-algebra is different from a $C^\star$-algebra. We suspect however the following conjecture.

Conjecture. The algebra $\ell^1(\Sigma)$ should be an hermitian Banach *-algebra.

For the further analysis we need to know more structure of this algebra. The first fact is the map $E$ defined as $E(a) = a(0)$. By definition, then, this map becomes a bi-module Banach space projection of norm one from $\ell^1(\Sigma)$ to $C(X)$. Furthermore, the automorphism $\alpha$ extends to an inner automorphism of $\ell^1(\Sigma)$ implimented by the unitary element $\delta$ for which $\delta(n) = \delta_n^1$.

It should be mentioned here that a serious difference between this $\ell^1$-crossed product and $C^\star$-crossed product,i.e. the existense of a closed ideals which is not selfadjoint even in the case $\ell^1(Z)$ is however clarified in the following way in the coming author's joint paper [2].

**Theorem 2.3** *For a dynamical system $\Sigma = \{X, \sigma\}$ , every closed ideal of $\ell^1(\Sigma)$ becomes selfadjoint if and only if $\Sigma$ is free, that is, no periodic points.*

We remark that the existence of a non-selfadjoint closed ideal shows that we can not expect another principle for the structure of closed ideals in $\ell^1(\Sigma)$ in general, that is, any closed ideal is the intersection of primitive ideals. For, an irreducible representation of $\ell^1(\Sigma)$ always extends to the irreducible representation of $C^*(\Sigma)$ and hence any primitive ideal of $\ell^1(\Sigma)$ becomes selfadjoint.

# References


[1] F.F.Bonsall and J.Duncan, Complete normed algebras, Springer,Berlin 1973

[2] M.de Jeu, C.Svensson and J.Tomiyama, On the Banach *-algebra crossed product associated with a topological dynamical system, preprint.

[3] T.W.Palmer, Banach algebras and the general theory of *-algebras, vol.II,Cambridge Univ. Press,2001

[4] W.Rudin, Fourier analysis on groups, Interscience Publishers, New York, London, 1962

[5] C.Svensson and J.Tomiyama, On the commutant of $C(X)$ in C*-crossed products by $Z$ and their representations, J.Funct.Analy.,256(2009) 2367-2386

[6] M.Takesaki, Theory of operator algebras I, Springer,New York, 1979

[7] J.Tomiyama, The interplay between topological dynamics and theory of C*-algebras, Lecture notes Ser.,vol.2, Res.Inst.Math., Seoul,1992

[8] J.Tomiyama, Hulls and kernels from topological dynamical systems and their applications to homeomorphism C*-algebras, J. Math. Soc. Japan, 56(2004),349-364

[9] J.Tomiyama, Classification of ideals of homeomorphism C*-algebras and quasidiagonality of their quotient algebras, Acta Appl. Math.,108 (2009),561-572. Proc. of Conf. Banff 2006.


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